Gromov Witten Invariants of K3 surfaces.

Joint with Rahul Pandharipande

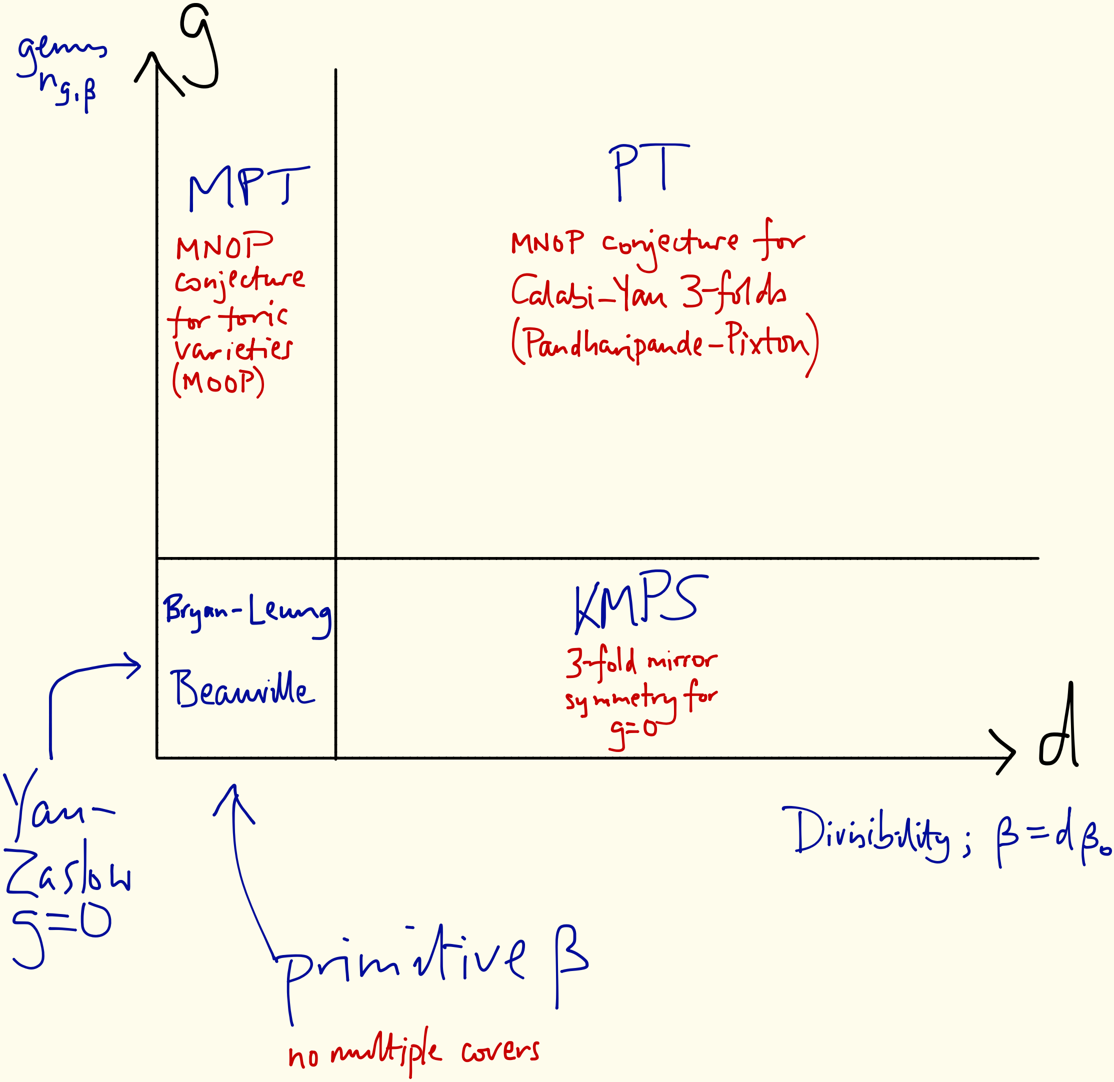
Prove **K**atz–**K**lemm–**V**afa conjecture expressing GW theory of K3 surfaces in terms of modular forms.

$$\sum_{g,h=0}^{\infty} (-1)^g n_{g,h} \left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^{2g} q^h$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-zq^n)^2(1-z^{-1}q^n)^2}$$

GV invts $n_{g,\beta}$ defined by universal formulae from GW invts $N_{g,\beta}$

$n_{g,\beta}$ depends only on h, where $\beta^2 = 2h-2$, and is integral.



GW too hard $\Rightarrow$ Use more linear theory: Stable pairs.

Gromov-Witten theory.

Holomorphic curves as (images of) (pseudo)holomorphic maps.

X smooth projective variety (symplectic manifold).

$\beta \in H_2(X, \mathbb{Z})$

$$M_g(X,\beta) = \left\{ \begin{array}{l} f: C \longrightarrow X \text{ holomorphic, } C \text{ nodal at worst,} \\ f_*[C] = \beta, \ |\mathrm{Aut}\, f| < \infty \end{array} \right\}$$

Proper Deligne-Mumford stack / compact singular orbifold

Has a virtual cycle of virtual dimension

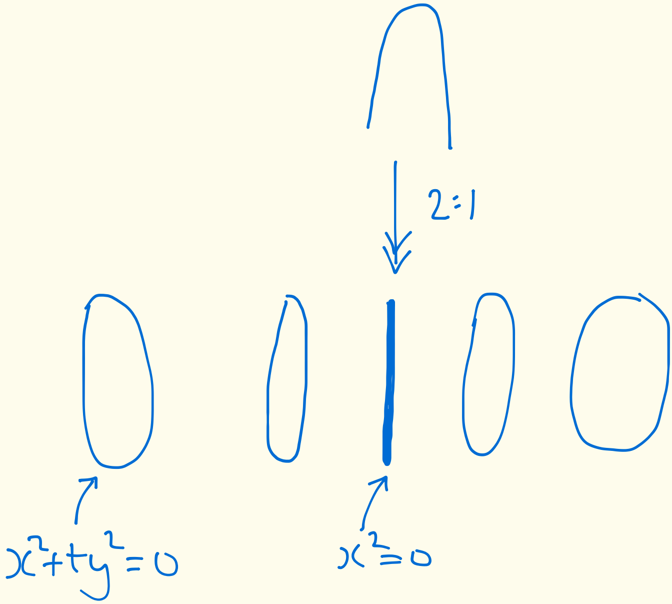
$$vd = \int_\beta c_1(X) + (\dim_{\mathbb{C}} X - 3)(1-g)$$

$\Rightarrow$ GW invariants

$$N_{g,\beta}(X, \ldots) = \int_{[M_g(X,\beta)]^{vir}} (\ldots)$$

$N_{g,\beta}(X) \in \mathbb{Q}$, deformation invariant

Eg conics in $\mathbb{P}^2$ degenerating to a double line



Limiting stable map double cover; counts as $1/2$.

$\exists$ underlying integer 1 counting the line in class $\beta/2$.

3 different ways of arranging these numbers:

$$Z^{GW}_{conn}(u,v) = \sum_{\substack{g\geq 0\\ \beta\neq 0}} N_{g,\beta}(X)\, u^{2g-2} v^{\beta}$$

$$= \sum_{\substack{g\geq 0\\ \beta\neq 0}} \underline{\underline{n_{g,\beta}}}\, u^{2g-2} \sum_{d>0} \frac{1}{d}\left(\frac{\sin du/2}{\sin u/2}\right)^{2g-2} v^{d\beta}.$$

Gopakumar–Vafa
BPS invariants,
Conjecturally in $\mathbb{Z}$
(Ionel–Parker, Bryan–Pandharipande)
Pandharipande–Pixton

$$\exp\left(Z^{GW}_{Conn}(u,v)\right) = \sum_{\beta\neq 0} Z^{GW}_{\beta}(u)\, v^{\beta},$$

where $$Z^{GW}_{\beta}(u) = \sum_{g=-\infty}^{\infty} N^{\bullet}_{g,\beta}(X)\, u^{2g-2}$$

Disconnected GW invariants
(No contracted connected components)

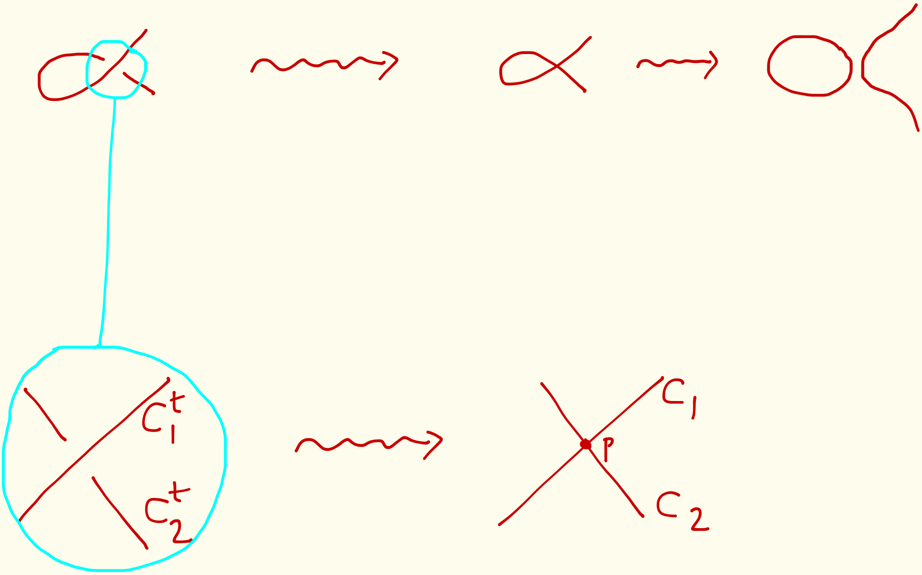
Stable Pairs

Want a theory counting embedded curves cut out by (holomorphic) equations — i.e. subschemes. Eg double line in conics example

<u>But</u> "genus change". $\underline{\underline{\dim_{\mathbb{C}} X = 3.}}$

$$\underset{g=0}{\text{twisted cubic}} \subset \mathbb{P}^3 \rightsquigarrow \underset{g=1}{\text{plane cubic}} \subset \mathbb{P}^2 \subset \mathbb{P}^3.$$

$$\mathbb{P}^1 \xrightarrow{t \mapsto} [1:t:t^2:t^3]$$



$C^t = C_1^t \cup C_2^t$ $\qquad$ $C = C_1 \cup C_2$

Using structure sheaves $\mathcal{O}_{C^t} = \mathcal{O}_{C_1^t} \oplus \mathcal{O}_{C_2^t}$

$\exists$ obvious flat limit $\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$ $\quad \left(\neq \mathcal{O}_C \text{ !!} \right)$

Not an abstract sheaf: has canonical section $(1,1)$.

Upshot is a stable pair

$$\mathcal{O}_X \xrightarrow{(1,1)} \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}.$$

Kernel $\mathcal{I}_C$
Cokernel $\mathcal{O}_p$

Defn: A stable pair (F, s) on X consists of:

1. F coherent sheaf, 1-dimensional support
2. $s \in H^0(F)$ section

Such that
(a) F is pure (has no subsheaves of dim 0)
(b) s has finite cokernel.

(F, s) has a support C

Pure dimension 1 subscheme
Cohen-Macaulay curve
No embedded points

and cokernel supported on a 0-dimensional subscheme $Z \subset C$.

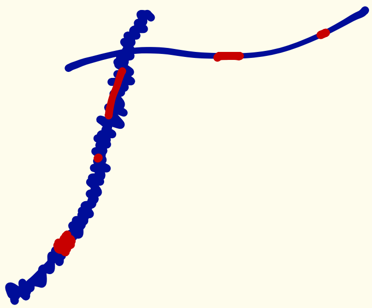
If C is Gorenstein then the space of pairs supported on C is $\mathrm{Hilb}^n C$.

Roughly, curve + line bundle + section.

Eg. $(\mathcal{O}_C, 1)$ bare curve C

$(\mathcal{O}_C(p_i), s_{p_i})$ C + Cartier divisor (p_i)

$(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}, (1,1))$ $C = C_1 \cup C_2$ + intersection points p_i.

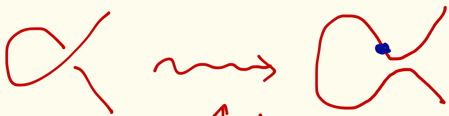


Invariants $\left([F], \chi(F)\right) = (\beta, n) \in H_2(X,\mathbb{Z}) \oplus \mathbb{Z}$

$\left(\text{Roughly } \left(ch_2(F),\ ch_3(F)\right)\right)$

If C reduced, $\underline{\chi(F) = 1 - g(C) + \#(\text{free pts})}$

Twisted cubic example;



$g \uparrow +1$

$\#\text{free pts} \uparrow +1$

Moduli space $P_n(X,\beta)$ projective scheme;
$\exists$ virtual cycle of virtual dimension $vd = \int_\beta c_1(X)$

$\Rightarrow$ Stable pair invariants $P_{n,\beta}(X, \ldots) = \int_{[P_n(X,\beta)]^{vir}} (\ldots)$

Integers; count curves + pts on them.

Generating function $Z_\beta^P(q) = \sum_{n \in \mathbb{Z}} P_{n,\beta}(X)\, q^n$

Maulik–Nekrasov–Okounkov–Pandharipande conjecture.

"These two theories contain the same information"

Counting parameterised curves $\longleftrightarrow$ Counting unparameterised curves

$\mathbb{Q}$ ⋒ $\qquad\qquad$ $\mathbb{Z}$ ⋒

MNOP conjecture:

The Laurent series $Z^P_\beta(q)$ is that of a rational function invariant under $q \longleftrightarrow q^{-1}$, $\quad$ eg $\frac{q}{(1+q)^2} = q - 2q^2 + 3q^3 - 4q^4 + \dots$

and

$$\boxed{Z^P_\beta(-e^{iu}) = Z^{GW}_\beta(u).}$$

$$\sum_{g=-\infty}^{\infty} N^\bullet_{g,\beta}(X)\, u^{2g-2} = \sum_{n=-\infty}^{\infty} P_{n,\beta}(X)\, q^n, \quad q = -e^{iu}.$$

$\exists$ relative, equivariant, ... versions, all proved for toric X by [MOOP], and with descendents by Pandharipande-Pixton.

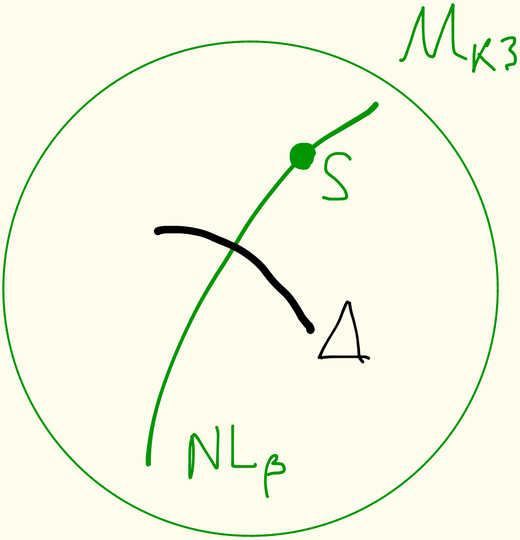
[PP] use this and degeneration to <u>prove</u> MNOP for "most" Calabi-Yau 3-folds (in particular complete intersections in toric varieties: enough for us).

Upshot is that to calculate GW invariants (inc. multiple covers) sufficient to calculate stable pairs invariants (inc. those supported on thickened curves).

K3 surfaces $\quad (S, \beta \in H_2(S, \mathbb{Z}))$

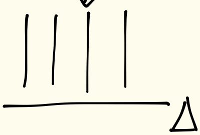
Can deform S so that $\beta \notin H^{1,1}(S)$

$$\Rightarrow \quad N_{g,\beta}(S) = 0 = P_{n,\beta}(S).$$



Instead use "twistor" K3-fibred CY 3-fold $X \to \Delta$, where Δ is transverse to NL_β (and NL_γ for $\gamma \leq \beta$)

$$\iota : S \hookrightarrow X$$



$(X, \iota_* \beta)$

KKV Conjecture concerns $N_{g,\iota_*\beta}(X)$

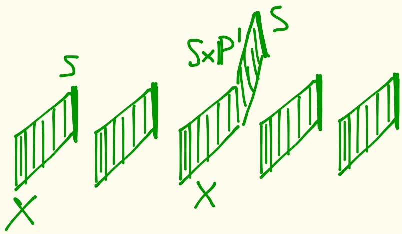
All curves in central fibre S (inc. multiple covers, etc.)

("Reduced GW invts of S with λ_g insertions")

By [PP]'s proof of MNOP sufficient to calculate stable pairs invariants of $(X, \iota_*\beta)$ $\forall \beta$ (inc. thickened curves).

<u>Method</u>: deform $(S \subset X)$ to $(S \subset N_{S/X})$, i.e. to $S \times \mathbb{C}$, by "deformation to normal cone of S".

$Bl_{S\times\{0\}}(X\times\mathbb{C})$



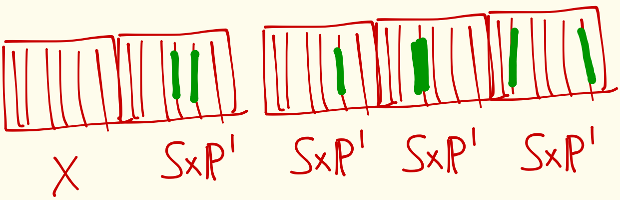
Jun Li (Li-Ruan, Ionel-Parker, SFT...)

$$\Rightarrow Z^P_{1_*\beta}(X) = Z^P_{1_*\beta}(X \cup_S S\times\mathbb{P}^1)$$

$$= Z^P(X/S) *_S Z^P(S\times\mathbb{P}^1/S\times\{0\})$$

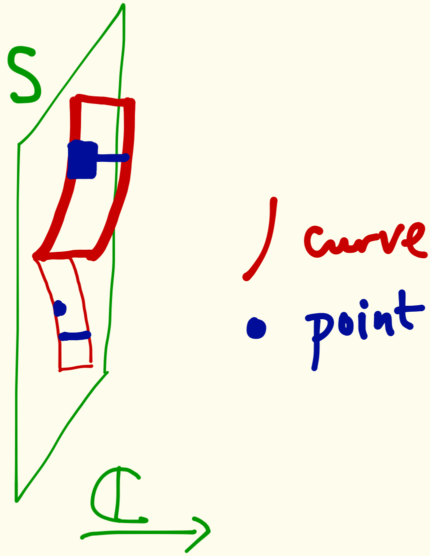
relative theories:
Curves intersect S in points; otherwise bubble off another $S\times\mathbb{P}^1$.

For us:

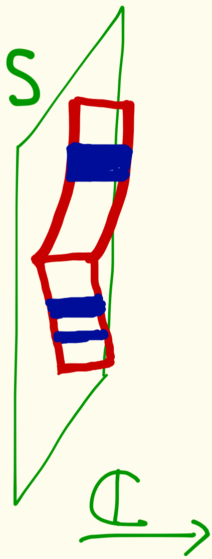


Localise with respect to $\mathbb{C}^*$ action on $S\times\mathbb{P}^1$

$\Rightarrow$ reduce to calculation with $\mathbb{C}^*$-fixed stable pairs on $S\times\mathbb{C}$ with nonstandard def. thy/virtual cycle.



By an extension of the argument that shows that the invariants of S (or $S\times\mathbb{C}$) vanish, show these vanish <u>unless</u> the pair is <u>universally</u> thickened in the $\mathbb{C}$ direction:



Moduli space of k-times thickened pairs is independent of k as a scheme (but not obstruction theory!)

$\Rightarrow$ Relate to $k=1$ case

$\Rightarrow$ Pairs on S

Moduli space $\mathrm{Hilb}^n\left(\mathcal{C}/|L|\right)$

$|L|$ - linear system of curves in class β

$\mathcal{C} \to |L|$ universal curve

Cut out of $|L| \times \mathrm{Hilb}^n S$ by incidence equations - section of a tautological bundle E.

Kool-T: this defines obstruction theory which coincides with <u>reduced</u> perfect obs. thy.

$\Rightarrow$ Virtual cycle is $e(E)$ (after push forward to $|L| \times \mathrm{Hilb}^n S$)

$\Rightarrow$ Can compute

Find result independent of divisibility of β

$\Rightarrow$ Can calc when β irreducible and moduli space smooth

This calc already done by <u>Kawai-Yoshioka</u>

Together with changes due to new obstruction theory this gives the result.